UNIVERSITY OF TWENTE.

Formal Methods & Tools.

A linear process algebraic format for probabilistic systems

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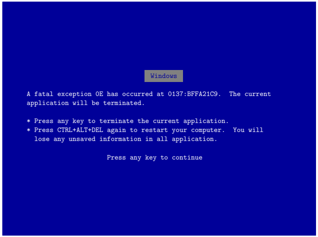
Joint work with Joost-Pieter Katoen, Jaco van de Pol, and Mariëlle Stoelinga

Contents

Introduction – Dependability

Dependability of computer systems is becoming more and more important.



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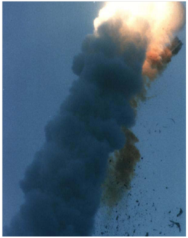

Ariane 5 crash

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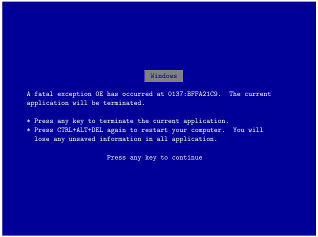

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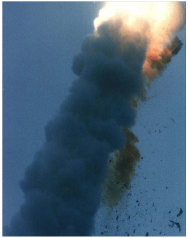
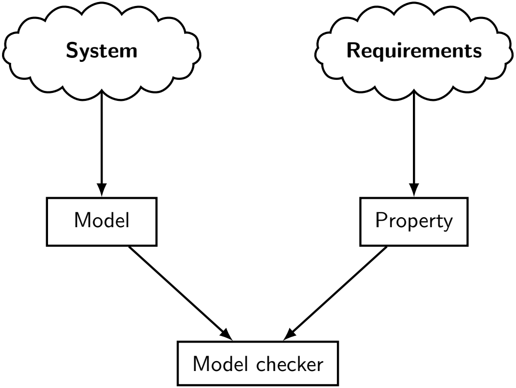


Ariane 5 crash

Our aim: use quantitative formal methods to improve system quality.

Introduction – Model Checking

A popular solution is model checking; verifying properties of a system by constructing a model and ranging over its state space.



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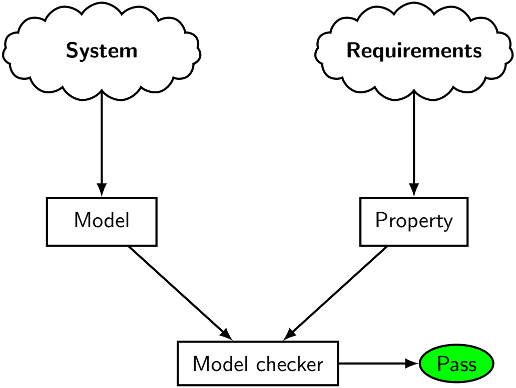

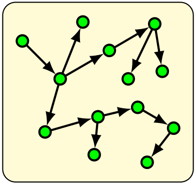

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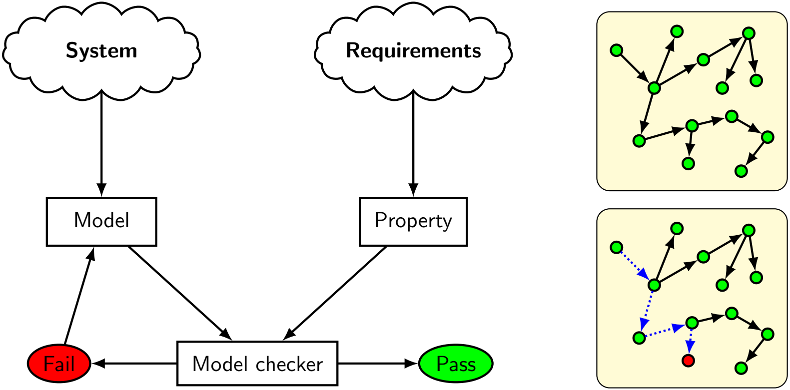

Introduction – Probabilistic Model Checking

Probabilistic model checking:

- Verifying quantitative properties,
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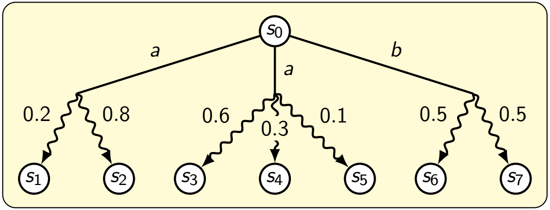

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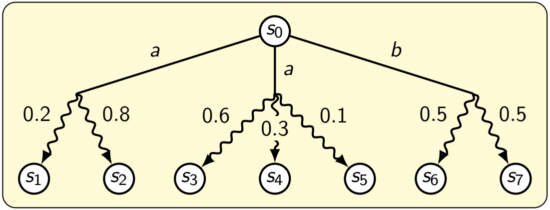

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Applications:

- Dependability analysis
- Performance analysis

Introduction – Probabilistic Model Checking

Limitations of previous approaches:

- Susceptible to the state space explosion problem
- Restricted treatment of data

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- Susceptible to the state space explosion problem
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Our approach:

1. Define a probabilistic process algebra (prCRL), incorporating both data types and probabilistic choice
2. Define a linear format (the LPPE), enabling symbolic optimisations at the language level
3. Develop and implement a linearisation algorithm
4. Reduce state spaces before they are generated by manipulations of the linear format.

Strong probabilistic bisimulation

Equivalent PAs: strong probabilistic bisimilar PAs

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Strong bisimulation

An *equivalence relation* R is a strong bisimulation if
$(p, q) \in R$ and $p \xrightarrow{a} p'$ imply that $q \xrightarrow{a} q'$ such that $(p', q') \in R$.

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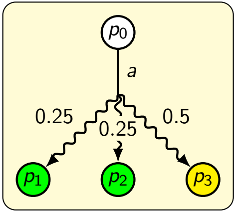

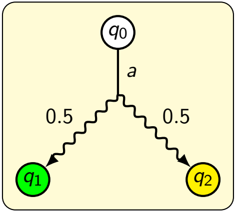

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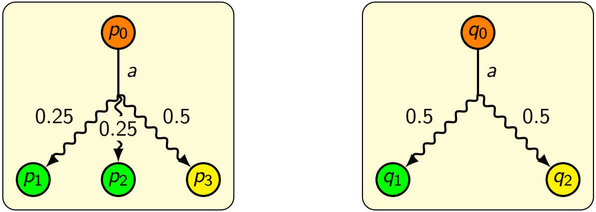
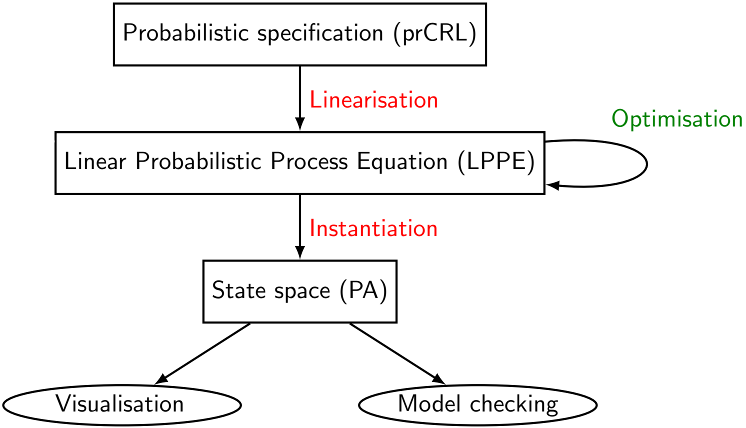


Introduction – overview of our approach



A process algebra with data and probability: prCRL

Specification language prCRL:

- Based on μCRL (so data), with additional probabilistic choice
- Operational semantics defined in terms of probabilistic automata
- Minimal set of operators to facilitate formal manipulation
- Syntactic sugar easily definable

A process algebra with data and probability: prCRL

The grammar of prCRL process terms

Process terms in prCRL are obtained by the following grammar:

$$p ::= Y(\vec{t}) \mid c \Rightarrow p \mid p + p \mid \sum_{x:D} p \mid a(\vec{t}) \sum\!\!\!\bullet_{x:D} f : p$$

- c is a condition (boolean expression)
- a is an atomic action
- f is a real-valued expression yielding values in $[0, 1]$
- $\vec{t}$ is a vector of expressions

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Process equations and processes

A process equation is something of the form $X(\vec{g} : \vec{G}) = p$.

Some examples

Sending an arbitrary natural number

$$X = \tau \sum_{n:\mathbb{N}^{>0}}^{\bullet} \frac{1}{2^n} : (\mathrm{send}(n) \cdot \sum_{j:\{*\}}^{\bullet} 1.0 : X)$$

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Sending ping messages until system crash

$$X = \mathsf{ping} \sum_{i:\{1,2\}}^{\bullet} (i = 1 \; ? \; 0.1 \; : \; 0.9) : ((i = 1 \Rightarrow \mathsf{crash}) + (i \neq 1 \Rightarrow X))$$

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Writing all Fibonacci numbers

$$X(p : \mathbb{N}, pp : \mathbb{N}) = \text{write}(\text{plus}(p, pp)) \cdot X(\text{plus}(p, pp), p)$$

Operational semantics

$$\text{NCHOICE-L} \; \frac{p \xrightarrow{\alpha} \mu}{p + q \xrightarrow{\alpha} \mu}$$

$$\text{IMPLIES} \; \frac{p \xrightarrow{\alpha} \mu}{c \Rightarrow p \xrightarrow{\alpha} \mu} \text{ if } c \text{ holds}$$

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An arbitrary specification

$$\begin{aligned} X(x : \text{Bool}) = x \quad & \Rightarrow a \cdot X(x) + b \cdot X(\text{not}(x)) \\ + \text{not}(x) & \Rightarrow c(0.5 : X(\text{false}) \oplus 0.5 : X(\text{true})) \end{aligned}$$

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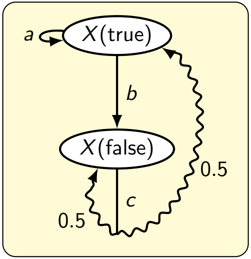

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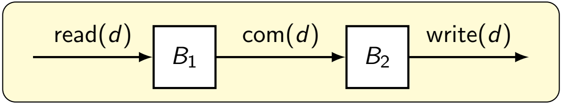
Linear process equations

In the non-probabilistic setting, LPEs are given by

$$
\begin{aligned}
X(\vec{g} : \vec{G}) = & \sum_{\vec{d_1}:\vec{D_1}} c_1 \Rightarrow a_1(b_1) \cdot X(n_1) \\
& \dots \\
+ & \sum_{\vec{d_k}:\vec{D_k}} c_k \Rightarrow a_k(b_k) \cdot X(n_k)
\end{aligned}
$$

- $\vec{G}$ is a type for state vectors
- $\vec{D_i}$ a type for local variable vectors for summand i
- c_i is the enabling condition of summand i
- a_i is an atomic action, with action-parameter vector b_i
- n_i is the next-state vector of summand i.

Linear process equations – An example



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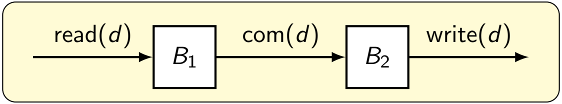

$$B_1 = \sum_{d:D} \mathrm{read}(d) \cdot \mathrm{com}(d) \cdot B_1$$

$$B_2 = \sum_{d:D} \overline{\mathrm{com}}(d) \cdot \mathrm{write}(d) \cdot B_2$$

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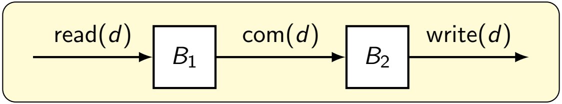

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$$\begin{array}{llll}
\multicolumn{4}{l}{X(a : \{1,2\}, b : \{1,2\}, x : D, y : D) =} \\
\sum_{d:D} & a = 1 & \Rightarrow \text{read}(d) \cdot X(2, b, d, y) & (1) \\
+ & a = 2 \wedge b = 1 & \Rightarrow \text{com}(x) \cdot X(1, 2, x, x) & (2) \\
+ & b = 2 & \Rightarrow \text{write}(y) \cdot X(a, 1, x, y) & (3)
\end{array}$$

A linear format for prCRL: the LPPE

In the probabilistic setting, LPPEs are given by

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Advantages of using LPPEs instead of prCRL specifications:

- Easy state space generation
- Straight-forward parallel composition
- **Symbolic optimisations enabled at the language level**

A linear format for prCRL: the LPPE

An example

$$X(\mathrm{pc} : \{1, 2\}, d : D) = \mathrm{pc} = 1 \Rightarrow \tau \sum_{e:D}\!\!\!\bullet\; \frac{1}{|D|} : X(2, e)$$
$$+ \mathrm{pc} = 2 \Rightarrow \mathrm{send}(d) \cdot X(1, d))$$

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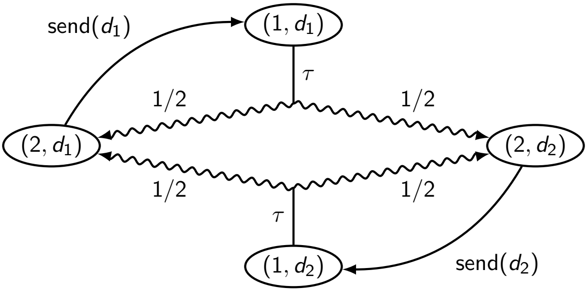

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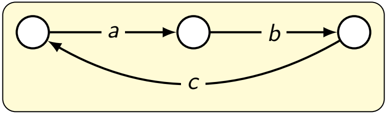

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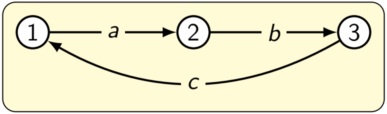

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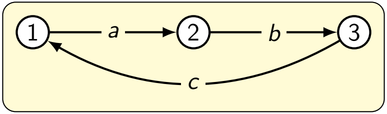

$$\begin{aligned} Y(pc\colon \{1,2,3\}) = & \\ & pc = 1 \Rightarrow a \cdot Y(2) \\ + & \; pc = 2 \Rightarrow b \cdot Y(3) \\ + & \; pc = 3 \Rightarrow c \cdot Y(1) \end{aligned}$$

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$$X = \sum_{d:D} \mathrm{in}(d) \cdot (\tau \cdot \mathrm{loss} \cdot X + \tau \cdot \mathrm{out}(d) \cdot X)$$

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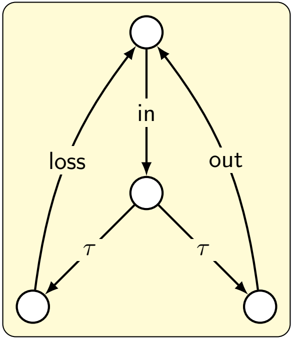

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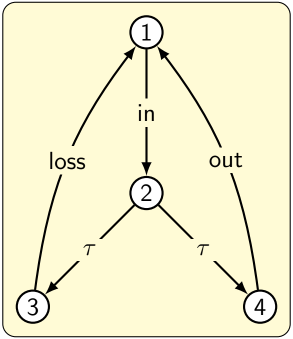

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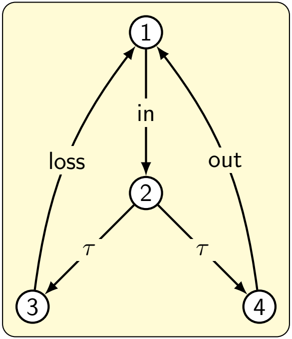

$$\begin{aligned} Y(pc\colon \{1,2,3,4\}, x\colon D) = \\ \textstyle\sum_{d:D} \quad & pc = 1 \Rightarrow \text{in}(d) \cdot Y(2, d) \\ + \quad & pc = 2 \Rightarrow \tau \cdot Y(3, x) \\ + \quad & pc = 2 \Rightarrow \tau \cdot Y(4, x) \\ + \quad & pc = 3 \Rightarrow \text{loss} \cdot Y(1, x) \\ + \quad & pc = 4 \Rightarrow \text{out}(x) \cdot Y(1, x) \end{aligned}$$

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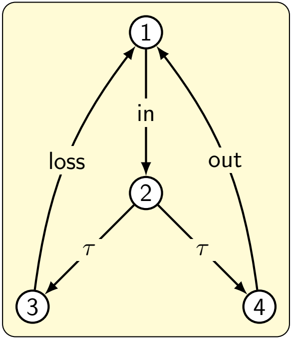

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Initial process: $Y(1, d_1)$.

Linearisation

$$X(d : D) = \sum_{e:D} a(d + e) \sum\!\!\!\!\bullet_{f:D} \frac{1}{|D|} : \left(c(e) \cdot c(f) \cdot X(5) + c(e + f) \cdot X(5) \right)$$

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---

2 $X_1(d : D, e : D, f : D) = \sum_{e:D} a(d + e) \sum\nolimits^{\bullet}_{f:D} \frac{1}{|D|} \cdot X_2(d, e, f)$

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$X_2(d : D, e : D, f : D) = c(e) \cdot c(f) \cdot X(5) + c(e+f) \cdot X(5)$

Linearisation

$$X(d:D) = \sum_{e:D} a(d+e) \sum_{f:D}\!\!\bullet\ \frac{1}{|D|} : \left(c(e)\cdot c(f)\cdot X(5) + c(e+f)\cdot X(5)\right)$$

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$$\begin{aligned} X(\mathrm{pc}:\{1,2,3\}, d:D, e:D, f:D) = \\ \mathrm{pc} = 1 \Rightarrow \sum_{e:D} a(d+e) \sum_{f:D}^{\bullet} \tfrac{1}{|D|} \cdot X(2,d,e,f) \\ + \mathrm{pc} = 2 \Rightarrow c(e) \cdot X(3,d,e,f) \\ + \mathrm{pc} = 2 \Rightarrow c(e+f) \cdot X(1,5,e,f) \\ + \mathrm{pc} = 3 \Rightarrow c(f) \cdot X(1,5,e,f) \end{aligned}$$

Linearisation

In general, we always linearise in two steps:

1. Transform the specification to intermediate regular form (IRF) (every process is a summation of single-action terms)
2. Merge all processes into one big process by introducing a program counter

In the first step, global parameters are introduced to remember the values of bound variables.

Theorem

A specification S and the specification S' obtained by linearising S are strongly probabilistic bisimilar.

Extended prCRL

For compositionality we introduce extended prCRL. It extends prCRL by parallel composition, encapsulation, hiding and renaming.

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The grammar of extended prCRL process terms

Process terms in extended prCRL are obtained by:

$$q ::= p \mid q \,||\, q \mid \partial_E(q) \mid \tau_H(q) \mid \rho_R(q)$$

- $q_1 \,||\, q_2$: parallel composition with ACP-style communication
- $\partial_E(q)$: encapsulation of all actions in E
- $\tau_H(q)$: hiding of all actions in H
- $\rho_R(q)$: renaming of actions according to the function R

Operational semantics of parallel composition

$$\textsc{Par-L}\ \frac{p \xrightarrow{\alpha} \mu}{p \,||\, q \xrightarrow{\alpha} \mu'} \text{ where } \forall p' \,.\, \mu'(p' \,||\, q) = \mu(p')$$

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$$X(d : D) = \mathsf{out}(d) \sum_{d' : D} \frac{1}{|D|} : X(d') \quad Y(n : \mathbb{N}) = \mathsf{write}(n) \cdot Y(n+1)$$

$$Z = X(d_1) \,||\, Y(0)$$

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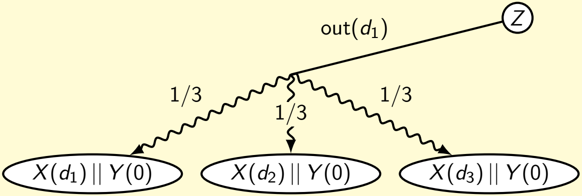

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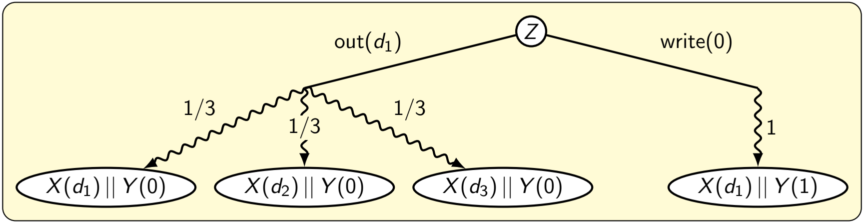

Operational semantics of parallel composition

For communication we assume a partial function

$$\gamma\colon \mathsf{Act} \times \mathsf{Act} \rightarrow \mathsf{Act}$$

When $(a, b) \in \mathrm{dom}(\gamma)$, communication between a and b yields $\gamma(a, b)$.

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$$X(n : \{2,3\}) = \text{write}(n) \cdot X(n) + c \sum_{n' : \{2,3\}} \frac{1}{2} : X(n')$$

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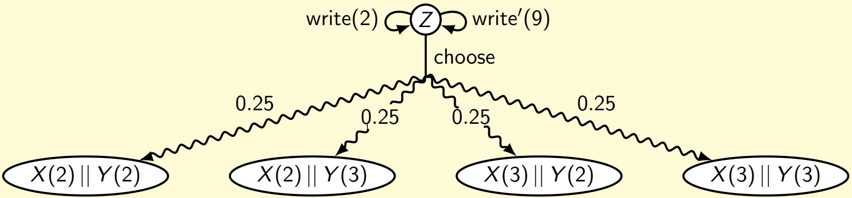

Linearisation of parallel composition

Linearisation of $X \,||\, Y$ is done compositionally: we first transform X and Y to LPPE, and then put them in parallel.

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$$\begin{aligned} Z(g, g') = &\sum_{d} c \Rightarrow a(b) \sum_{e}\!\!\bullet\; f : Z(n, g') \\ &+ \sum_{d'} c' \Rightarrow a'(b') \sum_{e'}\!\!\bullet\; f' : Z(g, n') \\ &+ \sum_{(d,d')} c \wedge c' \wedge b = b' \Rightarrow \gamma(a, a')(b) \sum_{(e,e')}\!\!\bullet\; f \cdot f' : Z(n, n') \end{aligned}$$

Linearisation of hiding, encapsulation and renaming

Linearisation of hiding, encapsulation and renaming is also done compositionally.

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X(g : G) \;=\; & \sum_{d_1 : D_1} c_1 \Rightarrow a_1(b_1) \sum\!\!\bullet_{e_1 : E_1} f_1 : X(n_1) \\
+ & \sum_{d_2 : D_2} c_2 \Rightarrow a_2(b_2) \sum\!\!\bullet_{e_2 : E_2} f_2 : X(n_2) \\
+ & \sum_{d_3 : D_3} c_3 \Rightarrow a_3(b_3) \sum\!\!\bullet_{e_3 : E_3} f_3 : X(n_3)
\end{aligned}
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Linearisation of hiding, encapsulation and renaming

Linearisation of hiding, encapsulation and renaming is also done compositionally.

$$\begin{aligned}\tau_{\{a_2\}}(X(g : G)) = & \sum_{d_1:D_1} c_1 \Rightarrow a_1(b_1) \bullet\!\!\!\!\sum_{e_1:E_1} f_1 : X(n_1) \\ + & \sum_{d_2:D_2} c_2 \Rightarrow a_2(b_2) \bullet\!\!\!\!\sum_{e_2:E_2} f_2 : X(n_2) \\ + & \sum_{d_3:D_3} c_3 \Rightarrow a_3(b_3) \bullet\!\!\!\!\sum_{e_3:E_3} f_3 : X(n_3)\end{aligned}$$

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Theorem

A specification S in extended prCRL and the specification S' obtained by linearising S are strongly probabilistic bisimilar.

Conclusions and Future Work

Conclusions / Results

- We developed the process algebra prCRL, incorporating both data and probability.
- We defined a linear format for prCRL, the LPPE, providing the starting point for effective symbolic optimisations and easy state space generation.
- We provided a linearisation algorithm to transform prCRL specifications to LPPEs, proved it correct, and implemented it.

Future work

Applying existing optimisation techniques to LPPEs

- constant elimination
- liveness analysis
- confluence reduction

Questions

Questions?